# Multistate Cohort Analysis with Proportional Transfer Rates: An Application to Marriage, Divorce, and Remarriage

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*ABSTRACT*

We apply a new approach to analyzing and synthesizing the behavior of a cohort as it proceeds through the life course and through different marital statuses. That approach overcomes the mathematical difficulties typically encountered in combining behavior over age intervals by assuming that the state-specific rates of movement are proportional over age. That Proportional Transfer Rate (PTR) assumption provides explicit mathematical relationships for the number of persons in each state at every age. The PTR approach is used to construct a 3-state, non-hierarchical, model of marriage, divorce, and remarriage. Current data for the United States indicate that the rates of first marriage, divorce, and remarriage are approximately proportional over age for women in 1995. In the PTR model, the stylized rates yield marital status trajectories over age, and facilitate analyses of the implications of changes in rate levels and patterns. A noteworthy finding is that contemporary high proportions married are quite sensitive to changes in first marriage rates, but less sensitive to changes in rates of divorce, and least sensitive to changes in rates of remarriage.

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Multistate life tables, which follow a cohort over time and reflect movements into and out of a specified set of states, have proven to be a valuable tool in demographic analyses (cf. Land and Rogers 1982; Schoen 1988ab). However, complexities in multistate calculations across age intervals, and difficulties in analytically representing state-specific trajectories with respect to size and relative composition, impede more extensive uses of the model.

Here, we build on the approach advanced in Schoen and Canudas-Romo (2005) that assumes that the rates of transfer vary uniformly over age (or time), preserving a fixed ratio between them. Under that Proportional Transfer Rate (PTR) assumption, calculations are significantly simplified, and the dynamics of the model are rendered more susceptible to analysis. We demonstrate those advantages in an analysis of cohort first marriage, divorce, and remarriage using data for the United States, 1995.

***The PTR Model***

Following Schoen and Canudas-Romo (2005), we set forth the general model with k living states. Let our base rate matrix $\boldsymbol{\mu}$ be a k x k matrix whose row i and column j element (i≠j) is $\mu_{ji}$, the base rate of transfer from state j to state i. When i=j, the jth diagonal element of $\boldsymbol{\mu}$ equals $-\sum \mu_{ji}$, where the sum over i ranges over all states except j. Let the PTR rate matrix for the x to x+n age interval, $\mathbf{M(x,n)}$, be given by

$$\mathbf{M(x,n)} = z(x,n)\,\boldsymbol{\mu} \qquad (1)$$

where z(x,n) is an age-interval specific function specifying the level of the transfer rates. The projection matrix that takes the cohort from age x to age x+n, $\mathbf{A(x,n)}$, is then

$$\mathbf{A(x,n)} = \exp[n\, \mathbf{M(x,n)}] = \exp[n\, z(x,n)\, \boldsymbol{\mu}] \tag{2}$$

Now let **y(x)** be a k element column vector describing the state composition of the cohort at age x, with jth element $y_j(x)$ representing the number of persons in state j at age x. By the usual projection relationship, we have

$$\mathbf{y(x+n)} = \mathbf{A(x,n)\, y(x)} \tag{3}$$

By virtue of the proportionality assumption in equation (1), the projection relationship can readily be extended over multiple age intervals. To project the population from age x to age x+w, we can use the product matrix

$$\mathbf{P(x,w)} = \mathbf{A(x+w\text{–}n,n)\, A(x+w\text{–}2n,n)} \ldots \mathbf{A(x,n)} \tag{4}$$

[Here we assume that w is an integral multiple of n, but that assumption can easily be relaxed.] As a result

$$\begin{aligned}\mathbf{y(x+w)} &= \mathbf{P(x,w)\, y(x)} \\ &= \exp[\ \{n \textstyle\sum z(a,n)\}\ \boldsymbol{\mu}\ ]\ \mathbf{y(x)}\end{aligned} \tag{5}$$

where the sum over a ranges from x to x+w–n. Equation (5) provides the PTR solution for the size and composition of the cohort at any age in terms of an initial population, base rate matrix **μ**, and the age-specific proportionality factors z.

***A PTR Model of Marriage, Divorce, and Remarriage***

Consider the base transfer rate matrix

$$\boldsymbol{\mu} = \begin{bmatrix} -\mu_{sm} & 0 & 0 \\ \mu_{sm} & -\mu_{mv} & \mu_{vm} \\ 0 & \mu_{mv} & -\mu_{vm} \end{bmatrix} \tag{6}$$

where subscript <u>s</u> indicates the Never Married state, subscript <u>m</u> the presently Married state, and subscript <u>v</u> the currently Divorced state. As shown in **Figure 1**, that matrix describes a three living state model where Never Married persons can marry, Married persons can divorce, and Divorced persons can remarry. To simplify the analysis, our focus is on ages 15 to 50, where mortality (and hence widowhood) can be ignored.

To implement equation (5), the base rate matrix in equation (6) can be written in terms of its eigenstructure (cf. Caswell 2001), i.e.

$$\boldsymbol{\mu} = \mathbf{U}\,\mathbf{R}\,\mathbf{V} \tag{7}$$

where $\mathbf{R}$ is a k x k diagonal matrix whose nonzero elements are the eigenvalues (roots) of $\boldsymbol{\mu}$, $\mathbf{U}$ is a k x k matrix whose jth column is the eigenvector (relative state composition) corresponding to the jth root, and $\mathbf{V}=\mathbf{U}^{-1}$, with the superscript (–1) denoting the matrix inverse. From matrix algebra (Gantmacher 1959), it follows that the population projection matrix for an n year interval is given by

$$\mathbf{A} = \mathbf{U}\exp(n\mathbf{R})\,\mathbf{V} = \mathbf{U}\,\boldsymbol{\Lambda}\,\mathbf{V} \tag{8}$$

where $\boldsymbol{\Lambda}= \exp(n\mathbf{R})$.

The eigenstructure of the base rate matrix in equation (6) can be written explicitly. Using Maple software and setting n=1, we find

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \exp[-\mu_{mv} - \mu_{vm}] & 0 \\ 0 & 0 & \exp[-\mu_{sm}] \end{bmatrix} \tag{9}$$

and

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & (\mu_{sm}-\mu_{mv}-\mu_{vm})/\mu_{mv} \\ \mu_{vm}/\mu_{mv} & -1 & (\mu_{vm}-\mu_{sm})/\mu_{mv} \\ 1 & 1 & 1 \end{bmatrix} \tag{10}$$

Now let our initial cohort population, **y(15)**, consist of one never married person, with no one currently married or divorced. Then, from equation (5), only the first column of **P(15,35)** influences **y(50)**. Using equations (9) and (10) we can write

$$\mathbf{y(50)} = \begin{bmatrix} \exp[-Z_{50}\,\mu_{sm}] \\ \\ \dfrac{\mu_{vm}}{\mu_{vm}+\mu_{mv}} - \dfrac{\exp[-Z_{50}(\mu_{vm}+\mu_{mv})]\mu_{sm}\,\mu_{mv}}{(\mu_{vm}+\mu_{mv}-\mu_{sm})(\mu_{vm}+\mu_{mv})} - \dfrac{(\mu_{vm}-\mu_{sm})\exp[-Z_{50}\,\mu_{sm}]}{\mu_{mv}+\mu_{vm}-\mu_{sm}} \\ \\ \dfrac{\mu_{mv}}{\mu_{vm}+\mu_{mv}} + \dfrac{\exp[-Z_{50}\,(\mu_{vm}+\mu_{mv})]\mu_{sm}\,\mu_{mv}}{(\mu_{vm}+\mu_{mv}-\mu_{sm})(\mu_{vm}+\mu_{mv})} - \dfrac{\mu_{mv}\exp[-Z_{50}\,\mu_{sm}]}{\mu_{mv}+\mu_{vm}-\mu_{sm}} \end{bmatrix} \qquad (11)$$

where $Z_{50}=\sum z(x)$, with that sum comprising the 35 age-specific z weights from 15 through 49. Since there is no mortality, the sum of the three elements of **y** is 1. The size and composition of the population at any age between 15 and 50 can be found from equation (11) by simply respecifying the sum of z(x) values.

Equation (11) can be integrated over age to find the number of person-years lived between any ages x and x+1. Let **Y(x)**, whose jth element $Y_j(x)$ refers to state j, represent the number of person-years lived at age x. Integrating over the x to x+1 interval yields

$$\mathbf{Y(x)} = \mathbf{U}\,[\,\mathbf{\Lambda}^{\mathbf{z(x)}} - \mathbf{I}\,]\,\mathbf{R}^{-1}\,\mathbf{V}\,\mathbf{y(x)} / z(x) \qquad (12)$$

where **I** is the k x k identity matrix. The elements of **Y** can be used to find the number of transfers between states in any age interval. For example, the number of moves from state i to state j between ages x+ and x+1 is given by $Y_i(x)$ times $z(x)\mu_{ij}$, the transfer rate from i to j at age x. The **Y** values can also be used in calculating population based life expectancies.

***An Application to Marriage and Divorce Data for the United States, 1995***

Data on marriage, divorce, and remarriage are available for the United States over much of the 20th century (Schoen 1987; Schoen and Standish 2001; see also Bumpass 1990; Cherlin 1992; Kreider and Fields 2001). Here we intend to use that data to follow cohort behavior under the PTR assumption. First, however, we need to determine whether the proportionality of rates assumption is tenable.

We examined age-specific U.S. rates of first marriage, divorce, and second or higher marriage for males and females in a number of periods and cohorts. While a strict proportionality did not hold, for the most part, the rates were approximately proportional (see **Figure 2**). The largest departures were with respect to rates based on relatively small populations, i.e. married persons under age 20 and divorced persons in their early and mid-20s.

To further examine the proportionality assumption, we started with the observed first marriage rates and calculated the relative sizes of the divorce and remarriage rates at ages 34-38. Those ages were chosen as they produced the closest agreement to the observed ratio of divorces to all marriages over the 15-49 age range. Scaling the base first marriage rates to 1, those relative sizes (1: 0.4491: 1.8701) were used as inputs to our base rate matrix $\boldsymbol{\mu}$. We then examined model cohort state distributions, by age, under (i) the observed rates and (ii) the "stylized" rates generated by $\boldsymbol{\mu}$ and the age pattern [z(x)] of the observed rates of first marriage.

The first two data columns of **Table 1** present the results for U.S. Females, 1995. They provide, for ages 25, 35, and 50, the proportion in each of the three marital states, as well as the proportion of all years lived between ages 15 and 50 in each state. Because the observed and stylized rates are identical for first marriages, the values for the Never Married State are the same. They are also quite similar for the Married and Divorced states. At age 50, where the fit is poorest, the proportions married differ by only 0.013 and the proportions divorced differ by only 0.007. The last row in Table 1 shows that the stylized and observed rates yield ratios of total divorces to total first marriages plus remarriages that differ by only 0.005. Given the patterns in Figure 2 and Table 1, we feel justified in our illustrative use of stylized rates.

*Sensitivity to Change in Z*

Sensitivity analyses can explore how state composition trajectories respond to changes in (i) the level of the rates, i.e. the values of z(x) and Z, and (ii) the relative pattern of the rates, i.e. the relative sizes of $\mu_{sm}$, $\mu_{mv}$, and $\mu_{vm}$. **Figure 3** shows how state composition under the U.S. 1995 stylized rates varies with Z, i.e. with the cumulative sum of the age-specific z(x) weights. As Z increases, the proportion Never Married decreases monotonically from 1 toward 0, while the proportion Married increases monotonically from 0 to 0.806, its long term (stable) value. Simultaneously, the proportion Divorced increases monotonically from 0 to 0.194, its long term value. Those stable values are almost attained when Z=6, as the proportion Never Married is 0.002, the proportion Married 0.805, and the proportion Divorced 0.193. The ultimate ratio of the proportion Married to the proportion Divorced is given by $\mu_{vm}/\mu_{mv}$, the row 2, column 1 element of the **U** matrix in equation (10). Here that ratio is 1.8701/0.4491 = 4.164.

The life course schematics shown in Figure 3 depend only on the pattern of the transfer rates in $\boldsymbol{\mu}$, not on their level. Changes in the magnitude of the transfer rates mean that the curves will be at a different value of Z, but their overall trajectories and relative sizes at any given value of Z will remain unchanged. Because $\mu_{sm}$ has been scaled to 1, Z represents the sum of the first marriage ages. At age 50, the PTR proportion Never Married is 0.116, hence $0.116=\exp(-Z_{50})$, or $Z_{50}=2.154$. In the neighborhood of that point, the shape of the state trajectories indicates how the proportion in each state would respond to a change in the magnitude of the transfer rates.

More rigorously, changes in the proportions in each state shown in equation (11) can be examined by differentiation with respect to Z. It is immediately clear that the proportion Never Married always decreases as Z increases. With $y_v$ representing the proportion Divorced, we have

$$dy_v/dZ = [\mu_{sm}\,\mu_{mv}\exp(-Z\{\mu_{mv}+\mu_{vm}\}) - \mu_{sm}\,\mu_{mv}\exp(-Z\mu_{sm})] \,/\, [\mu_{sm} - \mu_{mv} - \mu_{vm}] \qquad (13)$$

In general, there is no value at which that derivative is zero, indicating that the proportion Divorced monotonically increases to its stable value.

A similar analysis shows that the proportion Married, $y_m$, need not increase monotonically to stability. Differentiating, we find

$$dy_m/dZ = -[\mu_{sm}\,\mu_{mv}\exp(-Z\{\mu_{mv}+\mu_{vm}\}) + \mu_{sm}(\mu_{vm}-\mu_{sm})\exp(-Z\mu_{sm})]/[\mu_{sm} - \mu_{mv} - \mu_{vm}] \quad (14)$$

That derivative can be 0 for a positive value of Z if $\{\mu_{mv}+\mu_{vm}\}>\mu_{sm}>\mu_{vm}$. In that case, the value of $y_m$ "overshoots", in that it rises to a value greater than its ultimate stable value before decreasing monotonically to that stable level. With U.S. 1995 rates, the second inequality does not hold, hence the convergence to stability is monotonic.

*Sensitivity to Change in μ*

The impact of changes in transfer rates on state trajectories can be examined by differentiating the proportion in each state with respect to each rate. The results, however, are fairly complicated and rather uninformative. Instead, **Table 1** provides the proportions in each state at three specific ages and for the entire 15 to 50 age interval. Four scenarios are considered: (i) changes in first marriage rates, (ii) changes in divorce rates, (iii) changes in remarriage rates, and (iv) simultaneous changes in rates of divorce and remarriage. Each scenario is examined with rates increasing and decreasing 25% from their U.S. 1995 stylized values.

The values in Table 1 indicate that changes in first marriage rates have a substantial impact on the proportion in all 3 marital states. At age 50, a 25% increase in first marriage rates would lower the proportion Never Married from .116 to .068, while a 25% decrease in those rates would raise the proportion Never Married to.199. At the same time, an increase in first marriage rates substantially increases the proportions at age 50 Married (from .729 to .766) and Divorced (from .155 to .167). A decrease in first marriage rates has the opposite effect, reducing the proportions Married and Divorced. No other scenario has a greater effect on the proportions Married.

Increases in divorce rates raise the proportion Divorced and lower the proportion Married. This scenario has the greatest impact on the proportions Divorced, raising (or lowering) the proportion Divorced at age 50 by about 0.03. It also has the largest effect on the ratio of divorces to all marriages. Changing rates of both divorce and remarriage have somewhat offsetting effects, and have the smallest effect on proportions Married and Divorced at ages 35 and 50.

In sum, high rates of divorce play the major role in maintaining the sizeable proportions currently Divorced in the United States, as well as the high probability that a marriage will end in divorce. Nonetheless, the low U.S. proportions Never Married and high proportions Married are primarily the result of continuing high rates of first marriage.

***Conclusion***

The Proportional Transfer Rate model described here provides a potentially useful approach to cohort analysis. It can synthesize multistate patterns of cohort behavior, depict state composition trajectories over age, and facilitate analyses of the impact of changes in both the magnitude and the relative size of interstate transfer rates.

The life course schematics presented in Figure 3 show the pattern of trajectories by marital status implied by the base transfer rates, independent of their level. In the three state Never Married, Married, and Divorced model, the proportion Never Married always decreases monotonically to 0 as the level of the rates rises, while the proportion divorced always increases monotonically from 0 to its stable level ($\mu_{mv}$ /[$\mu_{mv}$+$\mu_{vm}$]). The proportion Married may increase monotonically from 0 to its stable level ($\mu_{vm}$/[$\mu_{mv}$+$\mu_{vm}$]), but in certain cases may overshoot that level and then monotonically approach it from above.

Changes in the relative levels of the base transfer rates can substantially affect the proportions by marital status. An analysis based on rates observed for U.S. Females, 1995 was conducted to gauge the effects of 25% increases and decreases in rates of first marriage, divorce, and remarriage. Changes in divorce rates had the greatest impact on the proportion Divorced at age 50, causing it to vary between 12% and 19%. However, changes in first marriage rates had the greatest effect on the other two marital statuses. At age 50, 25% changes in first marriage rates caused the proportion Never Married to vary from 7% to 20%, and the proportion Married to go from 77% to 66%.

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Figure 1. Diagram of a Multistate Model of First Marriage, Divorce and Remarriage.

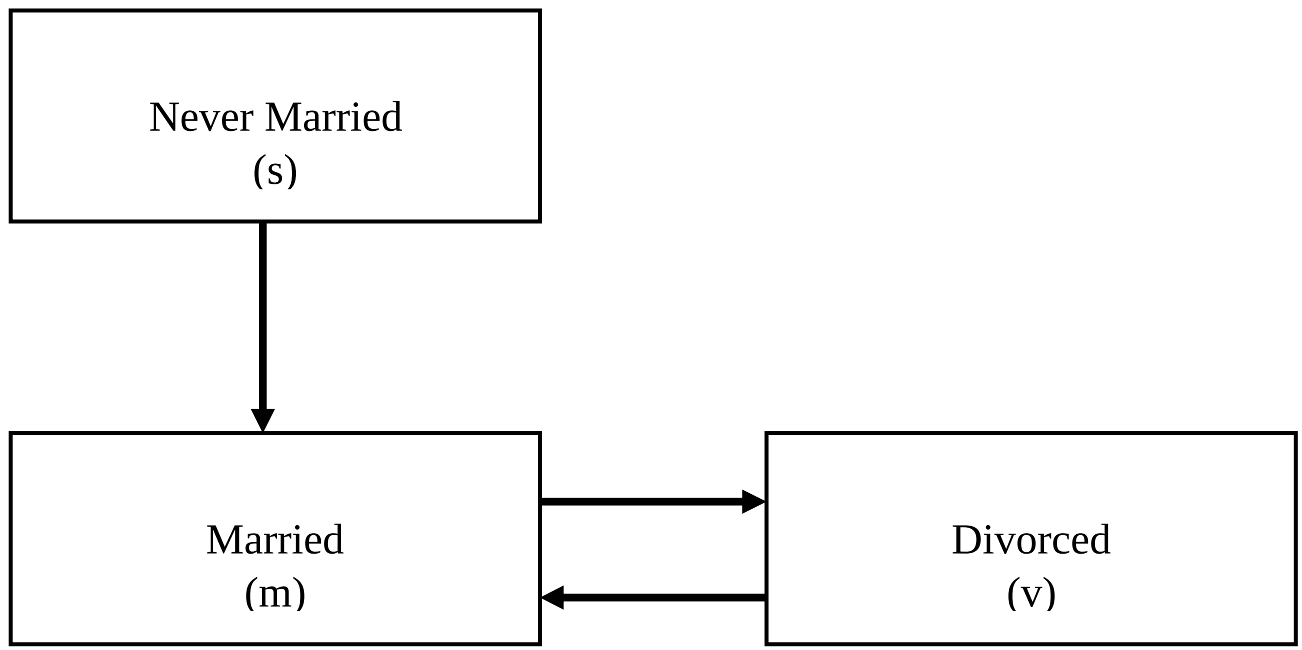

Figure 2. Observed (obs) and Stylized (sty) Rates of First Marriage, Divorce, and Remarriage with the Stylized Rates Based on Ages 34 to 38, United States Females, 1995.

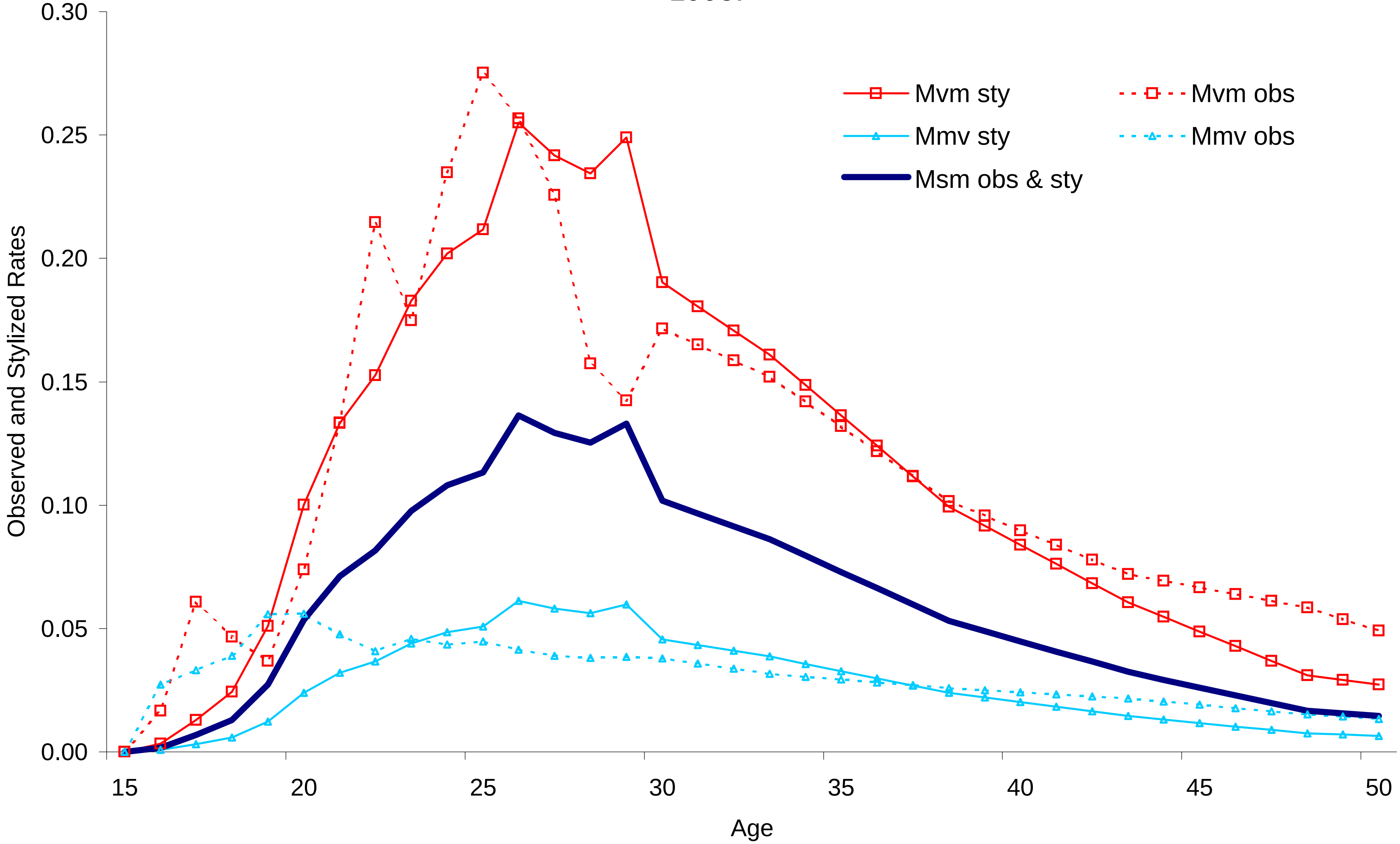

Figure 3. Life Course Schematics Showing Proportions Never Married, Married, and Divorced for Z Values From 0 to 6, United States Females, 1995.

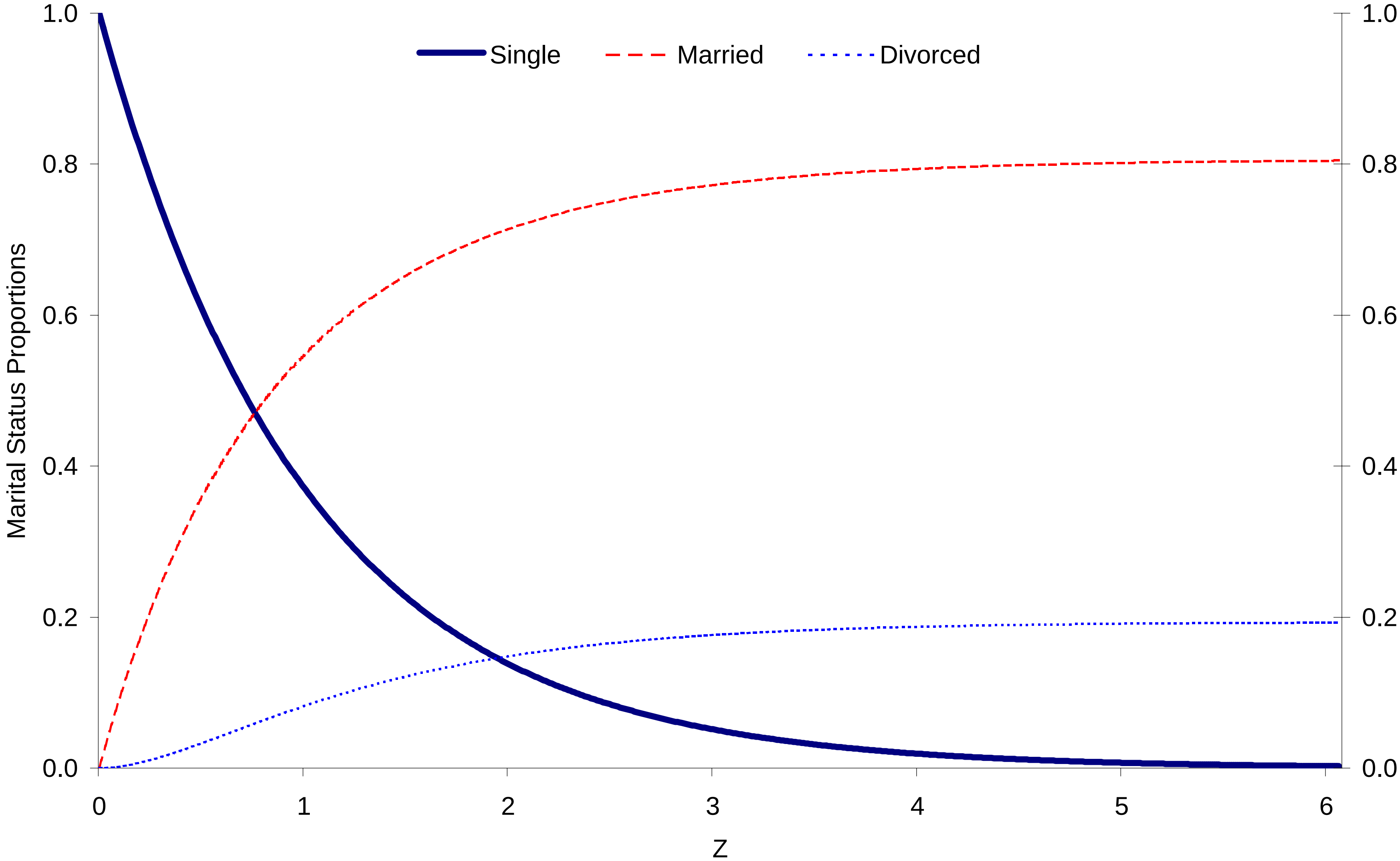

Table 1. The Impact of 25% Changes in the Base Pattern of Marital Status Rates on the Proportions of Never Married, Married, and Divorced Under Four Different Scenarios, PTR Model for US Females, 1995.

| | | | First Marriage Rates | | Divorce Rates | | Remarriage Rates | | Divorce and Remarriage | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Observed | Stylized | Increase | Decline | Increase | Decline | Increase | Decline | Increase | Decline |
| **Proportion of Persons in Marital Status by Age** | | | | | | | | | | |
| Never Married | | | | | | | | | | |
| 25 | 0.563 | 0.563 | 0.488 | 0.650 | 0.563 | 0.563 | 0.563 | 0.563 | 0.563 | 0.563 |
| 35 | 0.197 | 0.197 | 0.131 | 0.295 | 0.197 | 0.197 | 0.197 | 0.197 | 0.197 | 0.197 |
| 50 | 0.116 | 0.116 | 0.068 | 0.199 | 0.116 | 0.116 | 0.116 | 0.116 | 0.116 | 0.116 |
| Married | | | | | | | | | | |
| 25 | 0.399 | 0.396 | 0.464 | 0.318 | 0.387 | 0.406 | 0.399 | 0.393 | 0.390 | 0.403 |
| 35 | 0.688 | 0.673 | 0.725 | 0.593 | 0.647 | 0.702 | 0.691 | 0.651 | 0.667 | 0.684 |
| 50 | 0.716 | 0.729 | 0.766 | 0.664 | 0.698 | 0.763 | 0.752 | 0.697 | 0.724 | 0.737 |
| Divorced | | | | | | | | | | |
| 25 | 0.037 | 0.041 | 0.049 | 0.032 | 0.050 | 0.031 | 0.038 | 0.044 | 0.046 | 0.034 |
| 35 | 0.115 | 0.130 | 0.144 | 0.111 | 0.157 | 0.101 | 0.113 | 0.153 | 0.136 | 0.120 |
| 50 | 0.168 | 0.155 | 0.167 | 0.137 | 0.186 | 0.121 | 0.132 | 0.187 | 0.160 | 0.147 |
| **Proportion of Life in Each Status from 15 to 50** | | | | | | | | | | |
| Never Married | 0.374 | 0.374 | 0.319 | 0.453 | 0.374 | 0.374 | 0.374 | 0.374 | 0.374 | 0.374 |
| Married | 0.532 | 0.529 | 0.574 | 0.464 | 0.509 | 0.550 | 0.542 | 0.512 | 0.524 | 0.537 |
| Divorced | 0.094 | 0.097 | 0.107 | 0.083 | 0.117 | 0.075 | 0.084 | 0.114 | 0.102 | 0.089 |
| **Ratio of Total Divorces Over Total First Marriages and Remarriages** | | | | | | | | | | |
| | 0.418 | 0.413 | 0.425 | 0.400 | 0.470 | 0.344 | 0.410 | 0.418 | 0.467 | 0.347 |